

SHARP CAPACITY ESTIMATES IN S-JOHN DOMAINS

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ABSTRACT. It is well-known that several problems related to analysis on s -John domains can be unified by certain capacity lower estimates. In this paper, we obtain general lower bounds of p -capacity of a compact set E and the central Whitney cube Q_0 in terms of the Hausdorff q -content of E in an s -John domain Ω . Moreover, we construct several examples to show the essential sharpness of our estimates.

1. INTRODUCTION

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if there is a constant C and a point $x_0 \in \Omega$ so that, for each $x \in \Omega$, one can find a rectifiable curve $\gamma : [0, 1] \rightarrow \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and with

$$(1.1) \quad Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0, t]))$$

for each $0 < t \leq 1$. F. John used this condition in his work on elasticity [13] and the term was coined by Martio and Sarvas [18]. Smith and Stegenga [21] introduced the more general concept of s -John domains, $s \geq 1$, by replacing (1.1) with

$$(1.2) \quad Cd(\gamma(t), \partial\Omega) \geq l(\gamma([0, t]))^s.$$

The condition 1.1 is called a “twisted cone condition” in literature. Thus condition 1.2 should be called a “twisted cusp condition”.

In the last twenty years, s -John domains has been extensively studied in connection with Sobolev type inequalities; see [3, 11, 9, 14, 17, 21]. In particular, Buckley and Koskela [3] have shown that a simply connected planar domain which supports a Sobolev-Poincaré inequality is an s -John domain for an appropriate s . Smith and Stegenga have shown that an s -John domain Ω is a p -Poincaré domain, provided $s < \frac{n}{n-1} + \frac{p-1}{n}$. In particular, if $s < \frac{n}{n-1}$, then Ω is a p -Poincaré domain for all $1 \leq p < \infty$. These results were further generalized to the case of (q, p) -Poincaré domains in [11, 14, 17]. Recall that a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said to be a (q, p) -Poincaré domain if there exists a constant $C_{q,p} =$

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$C_{q,p}(\Omega)$ such that

$$(1.3) \quad \left(\int_{\Omega} |u(x) - u_{\Omega}|^q dx \right)^{1/q} \leq C_{q,p} \left(\int_{\Omega} |\nabla u(x)|^p dx \right)^{1/p}$$

for all $u \in C^\infty(\Omega)$. Here $u_{\Omega} = \int_{\Omega} u(x) dx$. When $q = p$, Ω is termed a p -Poincaré domain and when $q > p$ we say that Ω supports a Sobolev-Poincaré inequality.

The recent studies [1, 5, 7] on mappings of finite distortion have generated new interest in the class of s -John domains. In particular, uniform continuity of quasiconformal mappings onto s -John domains was studied in [4, 6].

The proofs for both types of problems rely on certain capacity estimates for subsets of s -John domains. To be more precise, for the problem related to Sobolev-Poincaré inequalities, one uses the idea of Maz'ya [19, 20] to reduce the problem to capacity estimates of the form

$$\text{Cap}_p(E, Q_0, \Omega) \geq \psi(|E|),$$

where Q_0 is the fixed Whitney cube containing the (John) center x_0 and E is an admissible subset of Ω disjoint from Q_0 ; for (1.3), $\psi(t) = Ct^{p/q}$, see also [8, 17]. Here, by admissible we mean that E is an open set so that $\partial E \cap \Omega$ is a smooth submanifold. As for the uniform continuity of quasiconformal mappings onto s -John domains, one essentially needs a capacity estimate of the form

$$\text{Cap}_n(E, Q_0, \Omega) \geq \psi(\text{diam } E),$$

where E is a continuum in Ω disjoint from the central Whitney cube Q_0 ; see [4]. Thus one could expect that a more general capacity estimate of the form

$$(1.4) \quad \text{Cap}_p(E, Q_0, \Omega) \geq \psi(\mathcal{H}_{\infty}^q(E))$$

holds in certain s -John domains Ω , where E is a compact set in Ω disjoint from the central Whitney cube Q_0 and $\mathcal{H}_{\infty}^q(E)$ is the Hausdorff q -content of E . We confirm this expectation by showing the following result.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an s -John domain. For $0 < \varepsilon < 1$, $1 \leq p \leq n$ and $q \geq s(n-1) + 1 - p + \varepsilon$, there exists a positive constant $C(n, p, q, s, \varepsilon)$ such that

$$(1.5) \quad \text{Cap}_p(E, Q_0, \Omega) \geq C(n, p, q, s, \varepsilon) \left(\mathcal{H}_{\infty}^q(E) \right)^{\frac{s(n-1)+1-p+\varepsilon}{q}},$$

whenever $E \subset \Omega$ is a compact set disjoint from Q_0 .

Remark 1.2. If $p = n$, $q = 1$ and $E \subset \Omega$ is a continuum, then (1.5) reduces to the estimate

$$\text{Cap}_n(E, Q_0, \Omega) \geq C(n, s, \varepsilon) (\text{diam } E)^{(n-1)(s-1)+\varepsilon}.$$

The restriction becomes $1 \geq (s-1)(n-1) + \varepsilon$, which is equivalent to $s \leq 1 + \frac{1-\varepsilon}{n-1}$. The range for s is essentially sharp, see [6].

If $q = n$, then (1.5) reduces to the estimate

$$\text{Cap}_p(E, Q_0, \Omega) \geq C(n, s, \varepsilon) |E|^{\frac{(n-1)s+1-p+\varepsilon}{n}}.$$

The restriction becomes $s \leq 1 + \frac{p-\varepsilon}{n-1}$. Note that

$$1 + \frac{p}{n-1} > \frac{n}{n-1} + \frac{p-1}{n}.$$

This implies that if $s < 1 + \frac{p}{n-1}$, then Ω is a p -Poincaré domain. The range for s is sharp, see [11].

The estimate in Theorem 1.1 is essentially sharp in the sense that the exponent of $\mathcal{H}_\infty^q(E)$ in (1.5) cannot be made strictly smaller than $\frac{s(n-1)+1-p}{q}$; see Example 4.1 below.

Our second result shows that the requirement $q \geq s(n-1) + 1 - p + \varepsilon$ is essentially sharp in the sense that there exists an s -John domain $\Omega \subset \mathbb{R}^n$ such that no estimate of the form as in (1.4) holds in Ω whenever $q < \min\{s(n-1) + 1 - p, n\}$. This is somewhat surprising since the estimate in (1.5) does not degenerate when $q < s(n-1) + 1 - p$.

Theorem 1.3. Fix $1 \leq p \leq n$. There exists an s -John domain $\Omega \subset \mathbb{R}^n$ such that there is a sequence of compact sets E_j in Ω with the following properties:

- Each E_j is disjoint from the central Whitney cube Q_0 ;
- $\mathcal{H}_\infty^q(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$, whenever $q < \min\{(n-1)s + 1 - p, n\}$.

It would be interesting to know whether one can obtain an estimate of the form as in (1.4) when $q = (n-1)s + 1 - p$.

When $q < \min\{(n-1)s + 1 - p, \log_2(2^n - 1)\}$, the s -John domain Ω constructed in Theorem 1.3 is in fact Gromov hyperbolic in the quasihyperbolic metric. This is very surprising, since it was proven in [4] that for all Gromov hyperbolic s -John domains Ω , an estimate of the form as in (1.4) holds when $p = n$, $q = 1$ and $E \subset \Omega$ is a continuum. Our example shows that one can not replace the assumption being a continuum by just being compact, and still obtain the estimate for all s -John domains. For definitions and examples of Gromov hyperbolic domains, we refer to the beautiful monograph [2].

2. PRELIMINARY RESULTS

For an increasing function $\tau : [0, \infty) \rightarrow [0, \infty)$ with $\tau(0) = 0$, we denote by \mathcal{H}_∞^τ the Hausdorff τ -content: $\mathcal{H}_\infty^\tau(E) = \inf \sum_i \tau(r_i)$, where the infimum is taken over all coverings of $E \subset \mathbb{R}^n$ with balls $B(x_i, r_i)$, $i = 1, 2, \dots$. When $\tau(t) = t^s$ for some $0 < s < \infty$, we write $\mathcal{H}_\infty^s = \mathcal{H}_\infty^\tau$.

For disjoint compact sets E and F in the domain Ω , we denote by $\text{Cap}_p(E, F, \Omega)$ the p -capacity of the pair (E, F) :

$$\text{Cap}_p(E, F, \Omega) = \inf_u \int_{\Omega} |\nabla u(x)|^p dx,$$

where the infimum is taken over all continuous functions $u \in W_{loc}^{1,p}(\Omega)$ which satisfy $u(x) \leq 0$ for $x \in E$ and $u(x) \geq 1$ for $x \in F$.

Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$. Then $\mathbb{W} = \mathbb{W}(\Omega)$ denotes a Whitney decomposition of Ω , i.e. a collection of closed cubes $Q \subset \Omega$ with pairwise disjoint interiors and having edges parallel to the coordinate axes, such that $\Omega = \cup_{Q \in \mathbb{W}} Q$, the diameters of $Q \in \mathbb{W}$ belong to the set $\{2^{-j} : j \in \mathbb{Z}\}$ and satisfy the condition

$$\text{diam}(Q) \leq \text{dist}(Q, \partial\Omega) \leq 4 \text{diam}(Q).$$

For $j \in \mathbb{Z}$ we define

$$\mathbb{W}_j = \{Q \in \mathbb{W} : \text{diam}(Q) = 2^{-j}\}.$$

The following lemma is well-known, see for instance [15, Lemma 2.8].

Lemma 2.1. Fix $1 \leq p < \infty$. Let B_1, B_2, \dots be balls or cubes in \mathbb{R}^n , $a_j \geq 0$ and $\lambda > 1$. Then

$$\left\| \sum a_j \chi_{\lambda B_j} \right\|_p \leq C(\lambda, n, p) \left\| \sum a_j \chi_{B_j} \right\|_p$$

3. MAIN PROOFS

Proof of Theorem 1.1. The proof is a combination of several well-known arguments; in particular [8, Proof of Theorem 9] and [11, Proof of Theorem 5.9]. For any compact set $E \subset \Omega$ such that $E \cap Q_0 = \emptyset$, where Q_0 is the central cube that contains the John center x_0 , we fix a test function u for $\text{Cap}_p(E, Q_0, \Omega)$, i.e. u is a continuous function in $W_{loc}^{1,p}(\Omega)$ so that $u \geq 1$ on E and $u \leq 0$ on Q_0 . We may assume that $\text{diam} \Omega = 1$.

For each $x \in E$, we may fix an s -John curve γ joining x to x_0 in Ω and define $P(x)$ to be the collection of Whitney cubes that intersect γ . Thus $Q(x) \in P(x)$ will be the Whitney cube containing the point x . We next divide our compact set E into the good part and the bad part according to the range of u_Q . Let $\mathcal{G} = \{x \in E : u_{Q(x)} \leq \frac{1}{2}\}$ and $\mathcal{B} = E \setminus \mathcal{G}$.

Claim 1: for $1 \leq p \leq n$ and $q \geq s(n-1) + 1 - p + \varepsilon$, there exists a positive constant $C(n, p, q, s, \varepsilon)$ such that

$$(3.1) \quad \int_{\Omega} |\nabla u(x)|^p dx \geq C(n, p, q, s, \varepsilon) \left(\mathcal{H}_{\infty}^q(\mathcal{B}) \right)^{\frac{s(n-1)+1-p+\varepsilon}{q}}.$$

Proof of Claim 1: Fix $1 \leq p \leq n$, $q \geq s(n-1) + 1 - p + \varepsilon$ and set $\Delta = \frac{\varepsilon}{2}$. Let $Q_i, i = 1, \dots, m$ be those Whitney cubes that intersect \mathcal{B} . Fix one such Whitney cube Q_{i_0} and let x_{i_0} be its center. Let $Q_{i_0}^j, j = 1, \dots, k$

be the Whitney cubes in $P(x_{i_0})$ with $Q_{i_0}^k = Q_{i_0}$. The standard chaining argument involving Poincaré inequality [21] gives us the estimate

$$1 \lesssim \sum_{j=1}^k \text{diam } Q_{i_0}^j \int_{Q_{i_0}^j} |\nabla u(y)| dy.$$

Hölder's inequality implies

$$1 \lesssim \left(\sum_{j=0}^k r_j^{(1-\kappa)p/(p-1)} \right)^{(p-1)/p} \left(\sum_{j=0}^k r_j^{\kappa p-n} \int_{Q_{i_0}^j} |\nabla u|^p \right)^{1/p},$$

where $r_j = \text{diam } Q_{i_0}^j$ and $\kappa = \frac{s+p-1-\Delta}{sp}$. Using the s -John condition, one can easily conclude

$$\sum_{j=0}^k r_j^{(1-\kappa)p/(p-1)} < C.$$

Therefore,

$$(3.2) \quad \sum_{j=0}^k r_j^{\kappa p-n} \int_{Q_{i_0}^j} |\nabla u|^p \geq C,$$

where the constant C depends only on p , n , Δ and the constant from the s -John condition.

By the s -John condition $Cr_j \geq |x-y|^s$, for $y \in Q_{i_0}^j$, and since $\kappa p - n < 0$ according to our choice $p \leq n$, we obtain

$$r_j^{\kappa p-n} \lesssim |x-y|^{s(\kappa p-n)}$$

for $y \in Q_{i_0}^j$. For $y \in Q_{i_0}^i \cap (2^{j+1}Q_{i_0} \setminus 2^jQ_{i_0})$, we have $|x-y| \approx 2^j r_k$ and hence for such y ,

$$(3.3) \quad r_i^{\kappa p-n} \lesssim (2^j r_k)^{s(\kappa p-n)}.$$

Combining (3.2) with (3.3) leads to

$$\begin{aligned} 1 &\lesssim \sum_{j=0}^k r_j^{\kappa p-n} \int_{Q_{i_0}^j} |\nabla u|^p \lesssim (r_k)^{s(\kappa p-n)} \int_{Q_{i_0}} |\nabla u|^p \\ &\quad + \sum_{j=0}^{|\log r_k|} (2^j r_k)^{s(\kappa p-n)} \int_{(2^{j+1}Q_{i_0} \setminus 2^jQ_{i_0}) \cap \Omega} |\nabla u|^p \\ &\lesssim \sum_{l=0}^{|\log r_k|+1} (2^l r_k)^{s(\kappa p-n)} \int_{2^l Q_{i_0} \cap \Omega} |\nabla u|^p. \end{aligned}$$

On the other hand,

$$\sum_{l=0}^{|\log r_k|+1} (2^l r_k)^\Delta < r_k^\Delta \sum_{l=-\infty}^{|\log r_k|+1} 2^{l\Delta} < C.$$

Combining the above two estimates, we conclude that there exists an l (depending on Δ and hence ε) such that

$$(2^l r_k)^\Delta \lesssim (2^l r_k)^{s(\kappa p - n)} \int_{2^l Q_{i_0} \cap \Omega} |\nabla u|^p.$$

It follows that,

$$\int_{\Omega \cap 2^l Q_{i_0}} |\nabla u|^p \gtrsim (2^l r_k)^{s(n - \kappa p) + \Delta} = (2^l r_k)^{s(n-1) + 1 - p + \varepsilon}.$$

In other words, there exists an $R_x \geq d(x, \partial\Omega)/2$ with

$$\left(\int_{\Omega \cap B(x, R_x)} |\nabla u|^p \right)^{\frac{q}{s(n-1) + 1 - p + \varepsilon}} \gtrsim R_x^q.$$

Applying the Vitali covering lemma to the covering $\{B(x, R_x)\}_{x \in E}$ of the set \mathcal{B} , we can select pairwise disjoint balls B_1, \dots, B_k, \dots such that $\mathcal{B} \subset \bigcup_{i=1}^\infty 5B_i$. Let r_i denote the radius of the ball B_i . Then

$$\begin{aligned} \mathcal{H}_\infty^q(\mathcal{B}) &\leq \sum_{i=1}^\infty (\text{diam } 5B_i)^q = 5^q \sum_{i=1}^\infty r_i^q \\ &\lesssim \sum_{i=1}^\infty \left(\int_{\Omega \cap B_i} |\nabla u|^p \right)^{\frac{q}{s(n-1) + 1 - p + \varepsilon}} \end{aligned}$$

The desired capacity estimate follows by noticing the elementary inequality

$$\sum_i a_i^b \lesssim \left(\sum_i a_i \right)^b, \quad b \geq 1.$$

Claim 2: for $n - q < p \leq n$ and $0 < \varepsilon < p + q - n$,

$$(3.4) \quad \int_{\Omega} |\nabla u(x)|^p dx \geq C(p, q, n, \varepsilon) \left(\mathcal{H}_\infty^q(\mathcal{G}) \right)^{\frac{n-p+\varepsilon}{q}}.$$

Proof of Claim 2: Fix $n - q < p \leq n$ and $0 < \varepsilon < p + q - n$. Our aim is to show that

$$(3.5) \quad \int_{2Q(x)} |\nabla u(x)|^p dx \geq C(p, s, n) \mathcal{H}_\infty^s(\mathcal{G} \cap Q(x))$$

for any $n - p < s \leq n$. We adapt the argument from [11, Proof of Theorem 5.9].

Fix $n - p < s \leq n$. For $y \in \mathcal{G}$, $u_{Q(y)} \leq \frac{1}{2}$. For $x \in \mathcal{G} \cap Q(y)$, write $Q_i = Q(x, r_i)$, where $r_i = 2^{-i-1} \text{diam } Q(y)$. Then

$$u(x) = \lim_{i \rightarrow \infty} u_{Q_i} = \lim_{i \rightarrow \infty} \int_{Q_i} u.$$

Now

$$\frac{1}{2} \leq |u(x) - u_{Q_0}| \leq \sum_{i \geq 0} |u_{Q_i} - u_{Q_{i+1}}|.$$

Since by the Poincaré inequality

$$|u_{Q_i} - u_{Q_{i+1}}| \leq C(n) r_i^{\frac{p+s-n}{p}} \left(r_i^{-s} \int_{Q_i} |\nabla u|^p \right)^{\frac{1}{p}},$$

we obtain that

$$\begin{aligned} \frac{1}{2} &\leq \sum_{i=1}^{\infty} C(n) r_i^{\frac{p+s-n}{p}} \left(r_i^{-s} \int_{Q_i} |\nabla u|^p \right)^{\frac{1}{p}} \\ &\leq C(p, s, n) (\text{diam } Q(y))^{\frac{p+s-n}{p}} \sup_{0 < t \leq \text{diam } Q(y)} \left(t^{-s} \int_{Q(x, t)} |\nabla u|^p \right)^{\frac{1}{p}} \\ &\leq C(p, s, n) \sup_{0 < t \leq \text{diam } Q(y)} \left(t^{-s} \int_{Q(x, t)} |\nabla u|^p \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, for each $x \in \mathcal{G} \cap Q(y)$, there is a cube $Q(x, t_x)$ such that $t_x \leq \text{diam } Q(y)$ and that

$$t_x^s \leq C(p, s, n) \int_{Q(x, t_x)} |\nabla u|^p.$$

By Vitali we can find pairwise disjoint cubes Q_1, Q_2, \dots as above such that $\mathcal{G} \cap Q(y) \subset \bigcup 5Q_i$. Then

$$\begin{aligned} \mathcal{H}_{\infty}^s(\mathcal{G} \cap Q(y)) &\leq C(p, s, n) \sum_{i=1}^{\infty} \int_{Q_i} |\nabla u|^p \\ &\leq C(p, s, n) \int_{2Q(y)} |\nabla u|^p. \end{aligned}$$

Thus the proof of (3.5) is complete.

We next show that for $n - q < p \leq n$ and for fixed $0 < \varepsilon < p + q - n$, the following estimate holds.

$$(3.6) \quad \int_{2Q(x)} |\nabla u(x)|^p dx \geq C(p, q, n, \varepsilon) \left(\mathcal{H}_{\infty}^q(\mathcal{G} \cap Q(x)) \right)^{\frac{n-p+\varepsilon}{q}}$$

Let $\varepsilon > 0$ be as above. We set $s = n - p + \varepsilon$. Then $s < q$. Now (3.6) follows from (3.5) and the trivial estimate

$$\left(\mathcal{H}_{\infty}^q(E) \right)^{\frac{s}{q}} \lesssim \mathcal{H}_{\infty}^s(E).$$

Taking into account the sub-additivity of Hausdorff q -content and concavity of the function $t \mapsto t^{\frac{n-p+\varepsilon}{q}}$, (3.4) follows immediately from (3.6) and Lemma 2.1.

□

4. EXAMPLES

Example 4.1. We will use the standard “rooms and corridors” type domains. This type of domains consists of a central cube shaped room along with an infinite disjoint collection of cube shaped rooms which are connected to the central room by narrow cylindrical corridors; see Figure 1.

For each $j \in \mathbb{N}$, the attached cube shaped room E_j is of edge length r_j and the narrow cylindrical corridor is of radius r_j^s and height r_j . We can ensure that the rooms and corridors are pairwise disjoint by requiring the sequence $\{r_j\}_{j \in \mathbb{N}}$ to decrease to zero sufficiently rapidly. It is clear that Ω is an s -John domain.

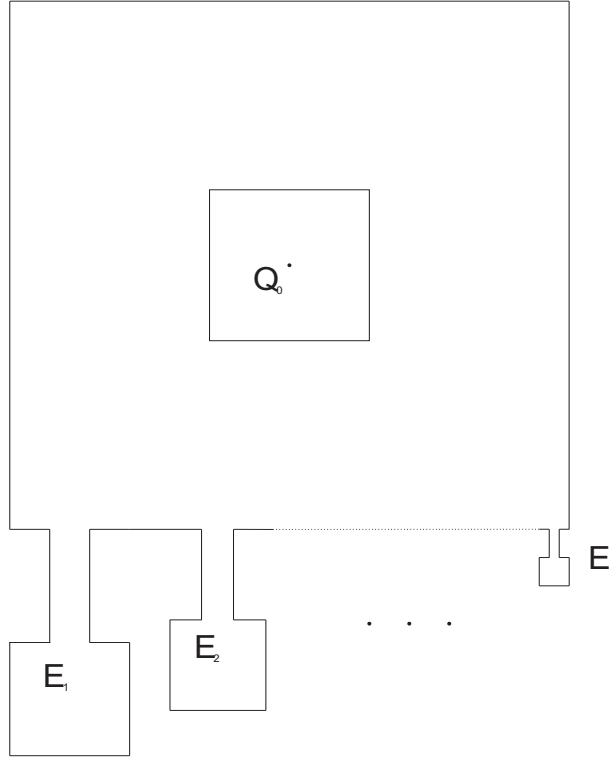


FIGURE 1. The standard “room and corridors” type domain

For $s < \frac{p+q-1}{n-1}$, we may choose $\varepsilon > 0$ such that $q \geq s(n-1) + 1 - p + \varepsilon$. Then it is easy to obtain the following estimate:

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq C r_j^{(n-1)s-p+1} \leq C \mathcal{H}_\infty^q(E_j)^{\frac{(n-1)s-p+1}{q}}$$

Noticing that $r_j \rightarrow 0$ as $j \rightarrow \infty$, this implies that the exponent of $\mathcal{H}_\infty^q(E)$ in Theorem 1.1 is essentially best possible.

Example 4.2. Fix $p \in [1, n]$, $n \geq 2$. There exists an s -John domain Ω in \mathbb{R}^n such that there is a sequence of compact sets E_j in Ω with the following two properties:

- Each E_j is disjoint from the central Whitney cube Q_0 ;
- $\mathcal{H}_\infty^q(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$, whenever $n - 1 \leq q < \min\{(n - 1)s + 1 - p, n\}$.

The idea of the construction of such an s -John domain is the following: we first construct a John domain Ω_0 such that the number N_j of Whitney cubes of size (comparable to) $r_j = 2^{-j}$ in Ω_0 is approximately 2^{qj} . We then build a “room and s -passage” Q_s in each Whitney cube $Q \subset \Omega_0$ and $Q \neq Q_0$, where Q_0 is the central Whitney cube containing the John center; see Figure 3. If the Whitney cube Q is of edge length $4r_j$, then the attached room shaped cube is of side length r_j and the corresponding s -passage is of radius r_j^s and height r_j .

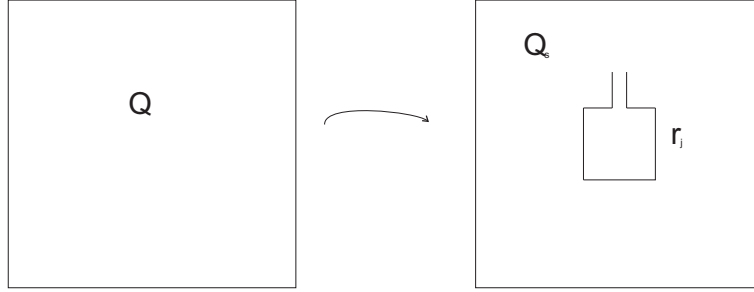


FIGURE 2. “room and s -passage” type replacement

Let E_j be the union of all the room shaped cube of edge length r_j . Then we have the following trivial upper estimate

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq CN_j \cdot r_j^{(n-1)s-p+1} \leq Cr_j^{(n-1)s-p-q+1}.$$

Thus $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ whenever $q < (n - 1)s - p + 1$. On the other hand, noting that all the cubes in E_j are well separated, to estimate the Hausdorff q -content, one has to cover each such cube by a ball of the same size (since otherwise the ball will intersects two cubes and substantially increases the radius). Thus we have

$$\mathcal{H}_\infty^q(E_j) \geq CN_j \cdot r_j^q \geq C.$$

To construct a John domain with the desired property, one essentially needs to construct a John domain Ω_0 such that $\dim_{\mathcal{M}}(\partial\Omega_0) = q$ when $q \in [n - 1, n)$, where $\dim_{\mathcal{M}}$ denotes the upper Minkowski dimension. With this understood, one can select certain Von Koch type curve as the boundary of a John domain; see [10, Proposition 5.2] for the detailed construction of such a John domain Ω_0 . It is clearly that the “room and s -passage” type replacement described above turns Ω_0 into an s -John domain Ω . In fact, $\dim_{\mathcal{M}}(\partial\Omega_0) = \dim_{\mathcal{M}}(\partial\Omega) = q$. For these facts, see [10, Proposition 5.11 and Proposition 5.16].

Example 4.3. Fix $1 \leq p \leq n$. There exists an s -John domain, which is Gromov hyperbolic in the quasihyperbolic metric, such that there is a sequence of compact sets E_j in Ω with the follow properties:

- Each E_j is disjoint from the central Whitney cube Q_0 ;
- $\mathcal{H}_\infty^q(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$, whenever $q < \min\{(n-1)s+1-p, \log_2(2^n-1)\}$.

We first give a detailed construction of the s -John domain Ω in the plane with the desired properties. Fix $1 \leq p \leq 2$. We first consider the case $q = \log_2 3$. The s -John domain Ω will be constructed by an inductive process. In the first step, we have a unit cube Q and four “room and s -passage” type “legs” as in Figure 3. The “ s -passage” R_1 is a rectangle of length 2^{-1} and width 2^{-s-1} and the “room” Q_1 is a cube of edge-length 2^{-1} . In the second step, we attach at each of the three corners of Q_1 a “room and s -passage” type “legs”. The “ s -passage” R_2 is a rectangle of length 2^{-2} and width 2^{-2s-1} and the “room” Q_2 is a cube of edge-length 2^{-2} . In general at step j , we have $4 \cdot 3^{j-1}$ “room and s -passage” type “legs”, where the “ s -passage” R_j is a rectangle of length 2^{-j} and width 2^{-js-1} and the “room” Q_j is a cube of edge-length 2^{-j} . It is easy to check that, with our choices of parameters, there is no overlap in our construction. Moreover, Ω is an s -John domain that is Gromov hyperbolic in the quasihyperbolic metric (since Ω is simply connected).

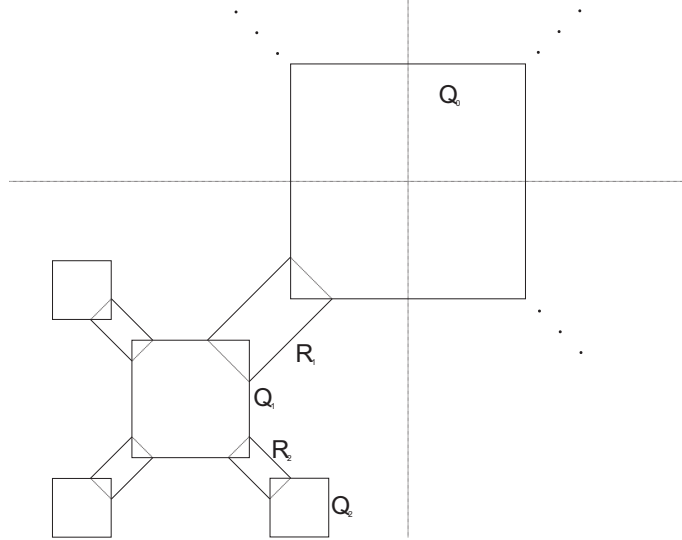


FIGURE 3. The s -John domain $\Omega \subset \mathbb{R}^2$

We choose E_j to be the union of all the cubes at step j , i.e. the collection of $4 \cdot 3^{j-1}$ (disjoint) cubes of edge-length 2^{-j} . Noting that all the cubes at step j are well separated, to estimate the Hausdorff q -content, one has to cover each such cube by a ball of the same size

(since otherwise the ball will intersect two cubes and substantially increases the radius). Note also that $q = \log_2 3$ and so it follows that

$$\mathcal{H}_\infty^q(E_j) \geq C4 \cdot 3^{j-1} \cdot 2^{-qj} = C.$$

On the other hand,

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq C4 \cdot 3^{j-1} \cdot 2^{-j(s-p+1)} \leq C2^{-j(s-p-q+1)}.$$

If $q < s - p + 1$, then $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$ as desired.

Next we consider the case $q < \min\{s - p + 1, \log_2 3\}$. This case is easier and we only need to delete some “room and s -passage” type “legs” from the previous construction. To be more precise, we choose $k_j \in \mathbb{N}$ to be an integer such that $k_j - 1 \leq 2^{qj} \leq k_j$. The construction of the desired s -John domain can be proceeded in a similar way. In the first step, we have a unit cube Q and k_1 “room and s -passage” type “legs” as in the previous construction. The “ s -passage” R_1 is a rectangle of length 2^{-1} and width 2^{-s-1} and the “room” Q_1 is a cube of edge-length 2^{-1} . In the second step, we fix k_2 corners of all the cubes of edge-length 2^{-1} in step 1, and attach at each corner a “room and s -passage” type “legs”. The “ s -passage” R_2 is a rectangle of length 2^{-2} and width 2^{-2s-1} and the “room” Q_2 is a cube of edge-length 2^{-2} . In general at step j , we have k_j “room and s -passage” type “legs”, where the “ s -passage” R_j is a rectangle of length 2^{-j} and width 2^{-js-1} and the “room” Q_j is a cube of edge-length 2^{-j} .

Let E_j be the union of all the cubes at step j , i.e. the collection of k_j (disjoint) cubes of edge-length 2^{-j} . It is clear that

$$\mathcal{H}_\infty^q(E_j) \geq Ck_j \cdot 2^{-qj} \geq C.$$

On the other hand, we have

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq Ck_j \cdot 2^{-j(s-p+1)} \leq C2^{-j(s-p-q+1)}.$$

If $q < s - p + 1$, then $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$ as desired.

We can construct similar examples in \mathbb{R}^n , $n \geq 3$. Fix $1 \leq p \leq n$. Consider the difficult case $q = \log_2(2^n - 1)$. The s -John domain Ω will be constructed in a similar manner as before. In the first step, we have a unit cube Q and 2^n “room and s -passage” type “legs”. The “ s -passage” R_1 is a cylinder of height 2^{-1} and radius 2^{-s-1} and the “room” Q_1 is a cube of edge-length 2^{-1} . In the second step, we attach at each of the $2^n - 1$ corners of Q_1 a “room and s -passage” type “legs”. The “ s -passage” R_2 is a cylinder of height 2^{-2} and radius 2^{-2s-1} and the “room” Q_2 is a cube of edge-length 2^{-2} . In general at step j , we have $2^n \cdot (2^n - 1)^{j-1}$ “room and s -passage” type “legs”, where the “ s -passage” R_j is a cylinder of height 2^{-j} and radius 2^{-js-1} and the “room” Q_j is a cube of edge-length 2^{-j} . It is easy to check that, with our choices of parameters, there is no overlap in our construction. Moreover, Ω is an

s -John domain that is Gromov hyperbolic in the quasihyperbolic metric. Indeed, one can easily verify that every quasihyperbolic geodesic triangle in Ω is δ -thin for some $\delta < \infty$.

We choose E_j to be the union of all the cubes at step j , i.e. the collection of $2^n \cdot (2^n - 1)^{j-1}$ (disjoint) cubes of edge-length 2^{-j} . Note that $q = \log_2(2^n - 1)$ and we obtain that

$$\mathcal{H}_\infty^q(E_j) \geq C 2^n \cdot (2^n - 1)^{j-1} \cdot 2^{-qj} = C.$$

On the other hand,

$$\begin{aligned} \text{Cap}_p(E_j, Q_0, \Omega) &\leq C 2^n \cdot (2^n - 1)^{j-1} \cdot 2^{-j[(n-1)s-p+1]} \\ &\leq C 2^{-j[(n-1)s-p-q+1]}. \end{aligned}$$

If $q < (n-1)s - p + 1$, then $\text{Cap}_p(E_j, Q_0, \Omega) \rightarrow 0$ as $j \rightarrow \infty$ as desired.

The case $q < \log_2(2^n - 1)$ can be proceeded as in the planar case by deleting the extra number of “room and s -passage” type “legs” and we leave the simple verification to the interested readers.

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